Reference Temperature Method for Computing Displacement Thickness

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THE purpose of the present note is to extend and simplify the calculation of a physical boundary-layer parameter, the displacement thickness, in high-speed flows where the properties of the external flow exhibit considerable variation inside the boundary layer. It is shown below that for a flat plate with zero pressure gradient an approximate method permits the calculation of the boundary-layer displacement thickness in closed form with good accuracy. The method can also be used for bodies of revolution if the Mangler transformation will be valid.

The specific need for a closed-form solution arises in cases where one deals with the growth problems of the boundary layer and the effect of the boundary layer on the adjacent external flow. One such problem is the hypersonic boundary-layer growth and the associated viscous interaction and pressure rise on bodies.[1-3] The success of the forementioned interaction methods rests on the assumption that the induced pressure can be obtained as a perturbation on the external flow (i.e., weak interaction) and is then determined by the local displacement thickness and its growth (slope), which, in turn, can be found from the zero pressure gradient boundary-layer solution. The latter two quantities can, in principle, be found from the solution of the system of compressible boundary-layer operations that are very difficult to solve, as all the fluid properties are variables. Even numerical methods require a number of restrictive assumptions and cause the already cumbersome interaction calculations to become more complicated and lengthy. Thus, the advantage of availability of closed form expressions cannot be overemphasized even if the resulting formulas are of approximative nature.

Presently available results for the compressible boundary-layer displacement thickness all seem to be based on a paper by Crocco,[4] with the most general result being due to Monaghan.[5] These methods make use of the following common assumptions: 1) flat-plate laminar boundary layer; 2) viscosity-temperature relationship is of the form $\mu/\mu_\infty = C(T/T_\infty)$; 3) Prandtl number is constant (often taken to be unity); and 4) c_p is constant.

The validity and applicability of the first three assumptions have been discussed in a number of places. The last assumption appears by necessity, as otherwise a numerical method would result. Although it seems to be valid for the particular experimental results supporting the analytical developments in viscous interaction,[2, 3] it may render the validity of the results at higher wall temperatures doubtful and in error. It has been found (for complete details see Ref. 6) that the fourth assumption can be removed by computing an effective c_p where the effect of variable c_p appears as a correction factor that need be computed only once for a particular boundary-layer calculation.

As the point of departure and the "exact" result to be improved upon, consider the Monaghan solution for the boundary-layer displacement thickness[5]:

$$\frac{\delta^*}{2x}(R_e)^{1/2} = \frac{C^{1/2}}{0.664}\left\{\int_0^1 \frac{c_{p_\infty}(A - BZ - DZ^2)}{c_p(1 - Z^2)^{1/2}}\,dZ - 1\right\} \quad (1)$$

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where C is the Chapman-Rubesin constant, $Z = U/U_\infty$,

$$A - BZ - DZ^2 = \frac{h}{h_\infty} = \frac{c_pT}{c_{p\infty}T_\infty}$$

with

$$A = \frac{h_w}{h_{aw}} \qquad B = Pr^{1/3}\frac{h_w - h_{aw}}{h_\infty}$$

$$D = Pr\left(\frac{\gamma - 1}{2}\right)M_\infty^2$$

Assuming that c_p = const = $c_{p\infty}$, Eq. (1) can be immediately integrated to give

$$\frac{\delta^*}{2x}(R_e)^{1/2} = \frac{C^{1/2}}{0.664}\left[\left(A - \frac{D}{2}\right)\frac{\pi}{2} - (B + 1)\right]$$

In the present development, for reasons explained previously, this assumption will not be used. Instead, it will be written

$$\frac{\delta^*}{2x}(R_e)^{1/2} = \frac{C^{1/2}}{0.664}\left\{\frac{c_{p_\infty}}{\bar{c}_p}\int_0^1 \frac{A - BZ - DZ^2}{(1 - Z^2)^{1/2}}\,dz - 1\right\}$$

which will integrate to yield

$$\frac{\delta^*}{2x}(R_e)^{1/2} = \frac{C^{1/2}}{0.664}\left\{\frac{c_{p_\infty}}{\bar{c}_p}\left[\left(A - \frac{D}{2}\right)\frac{\pi}{2} - B\right] - 1\right\}$$

Here $\bar{c}_p$ is an effective reference quantity evaluated at a suitably chosen reference temperature. The reference temperature cannot be determined in a rigorous manner; at best it can be considered to be a curve fit to previous solutions of equations of motion.[7]

The purpose of establishing any reference temperature has been, and also is in this case, to conveniently account for the fluid properties variation when the compressibility effects appear. The method cannot give the local boundary-layer properties, but it predicts with acceptable accuracy the (integrated) quantities like skin friction, heat transfer, etc.

For example, in Eckert's reference temperature method, all property variations (μ, Pr, R_e) are accounted for in an indirect way by retaining the incompressible form for the friction coefficient $c_f(R_e{}^*)^{1/2} = 0.664$ and evaluating the Reynolds number at the reference temperature[7]

$$T^*/T_\infty = 0.5(1 + A) + 0.22D/Pr^{1/2} \quad (2)$$

Another method is to include the viscosity temperature variation by a modified form of the Sutherland formula leading to the following expression for the skin friction:

$$c_f(R_e)^{1/2} = 0.664\,(C)^{1/2}$$

where the Chapman-Rubesin constant C can again be evaluated at a reference temperature. Two examples follow, the first due to Young[8] and the second due to Monaghan[5]:

$$T'/T_\infty = A - 0.468\,B - 0.273\,D \quad (3)$$

$$T'/T_\infty = 0.45 + 0.55\,A + 0.18\,D/Pr^{1/2} \quad (4)$$

where A, B, D already have been defined previously. In Eqs. (2–4), c_p has been assumed to be constant. In the formulation of these equations, the variation of the physical properties that are functions of temperature (μ, c_p, k, ρ) has been included, at least on the average, since these properties have been obtained empirically from the compressible boundary-layer solutions.

Therefore, it now seems reasonable to suppose, since the reference temperature predicts successfully an effective Reynolds number (and also an effective Prandtl number), that the same reference temperature could also be used to determine an effective specific heat $\bar{c}_p(T')$ or $\bar{c}_p(T^*)$.

Extensive numerical calculations indicate that the forementioned plausibility argument is indeed valid[6] and that

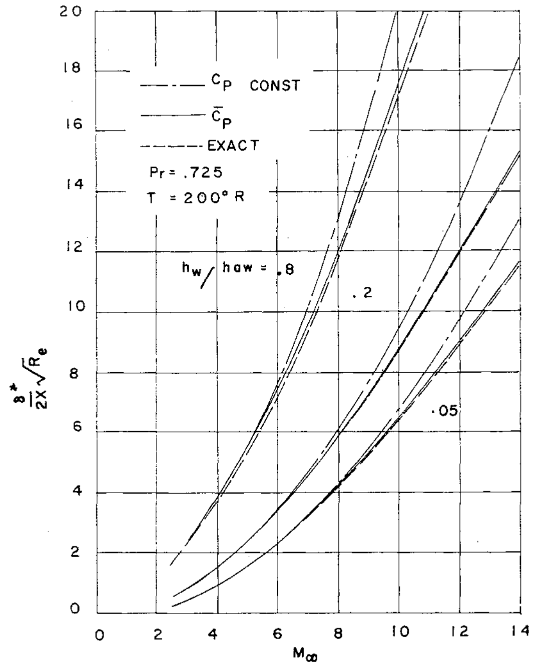

Fig. 1 Displacement thickness variation with Mach number.

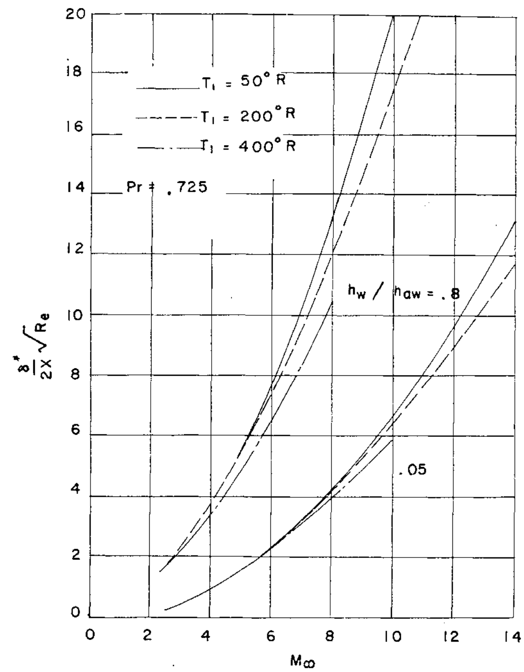

Fig. 2 Displacement thickness variation with Mach number.

any one of the reference temperatures indicated previously [Eqs. (2–4)] will determine $\bar{c}_p$ with satisfactory accuracy. Thus, the reference temperature concept, previously utilized only for determining skin friction and heat transfer, is also useful for computing other boundary-layer parameters.

The perfect gas computations were performed for the following range of the parameters:

$$0 \leq M \leq 14 \qquad 50 \leq T_\infty \leq 400°\text{R}$$

$$0.05 \leq h_w/h_{aw} \leq 2.0 \qquad 0.7 \leq Pr \leq 1.0$$

The effective specific heat was obtained from current tables[9, 10] by use of a reference temperature [Eq. (5) seemed to give consistently the best results], and the expression

$$\frac{c_{p_\infty}}{c_p}\left[\left(A - \frac{D}{2}\right)\frac{\pi}{2} - B\right]$$

was compared with the integral in Eq. (1). The average error was less than 2% (the maximum 4.3%) including the contribution from the integral in Eq. (1) due to its singular behavior as $Z \rightarrow 1$. The Prandtl number variation was found to be negligible.

The displacement thickness based on the reference temperature was also evaluated. Typical results are shown in Figs. 1 and 2. These show that, for low freestream temperatures, constant c_p assumption gives excellent results and deviations do not occur until about Mach number 10 has been reached, assuming that the ratio h_w/h_{aw} is less than about 1.0. Increasing the freestream temperature leads to considerable disagreement between the constant c_p assumption and the more accurate reference temperature method. Additional results show[6] that for $T_\infty = 400°$ the constant c_p method will be in error about 10% for $M = 8$ and $h_w/h_{aw} =$ 0.05. Furthermore, if h_w/h_{aw} is increased to 1.0, the same error occurs at Mach number 3.5. The results indicate that, for realistic computations at higher Mach number, c_p cannot be assumed to be constant, and its variation could be accounted for via the reference temperature method.

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